Série n ° 3 d'exercices corrigés sur « limites et continuité » **2éme Bac SM**

Exercice 1 :

Soit $a \in \mathbb{R}^*$ et $n \in \mathbb{N}^*$; on considère la fonction g_n définie par : $g_n(x) = \dfrac{1 - \prod_{k=1}^{n} \cos^k(kx)}{1 - \cos(ax)}$

On pose $U_n = \lim_{x \to 0} g_n(x)$

1) Montrer que $(\forall n \in \mathbb{N}^*)$; $U_{n+1} = U_n + \dfrac{(n+1)^3}{a^2}$

2) Calculer U_n en fonction de n et de a.

Correction

1) Soit $n \in \mathbb{N}^*$; on a : $U_{n+1} - U_n = \lim_{x \to 0} g_{n+1}(x) - \lim_{x \to 0} g_n(x)$

$$= \lim_{x \to 0}\left(\frac{1 - \prod_{k=1}^{n+1} \cos^k(kx)}{1 - \cos(ax)}\right) - \lim_{x \to 0} g_n\left(\frac{1 - \prod_{k=1}^{n} \cos^k(kx)}{1 - \cos(ax)}\right)$$

$$= \lim_{x \to 0}\left(\frac{1 - \prod_{k=1}^{n+1} \cos^k(kx)}{1 - \cos(ax)} - \frac{1 - \prod_{k=1}^{n} \cos^k(kx)}{1 - \cos(ax)}\right)$$

$$= \lim_{x \to 0}\left(\frac{-\cos^{n+1}((n+1)x)\prod_{k=1}^{n} \cos^k(kx) + \prod_{k=1}^{n} \cos^k(kx)}{1 - \cos(ax)}\right)$$

$$= \lim_{x \to 0}\left(\prod_{k=1}^{n} \cos^k(kx) \times \frac{1 - \cos^{n+1}((n+1)x)}{1 - \cos(ax)}\right)$$

Or $1 - \cos^{n+1}((n+1)x) = \left(1 - \cos((n+1)x)\right)\sum_{k=0}^{n} \cos^k((n+1)x)$

$$Donc\ U_{n+1}-U_n=\lim_{x\to 0}\left(\prod_{k=1}^{n}\cos^k(kx)\times\frac{\left(1-\cos((n+1)x)\right)\times\sum_{k=0}^{n}\cos^k((n+1)x)}{1-\cos(ax)}\right)$$

$$=\lim_{x\to 0}\left(\prod_{k=1}^{n}\cos^k(kx)\times\sum_{k=0}^{n}\cos^k((n+1)x)\times\frac{(n+1)^2}{a^2}\times\frac{\left(1-\cos((n+1)x)\right)}{((n+1)x)^2}\times\frac{(ax)^2}{1-\cos(ax)}\right)$$

$$Et\ \lim_{x\to 0}\frac{\left(1-\cos((n+1)x)\right)}{((n+1)x)^2}=\frac{1-\cos(ax)}{(ax)^2}=\frac{1}{2}$$

$$Donc\ \lim_{x\to 0}\left(\frac{\left(1-\cos((n+1)x)\right)}{((n+1)x)^2}\times\frac{(ax)^2}{1-\cos(ax)}\right)=1\ ;$$

$$\lim_{x\to 0}\prod_{k=1}^{n}\cos^k(kx)\times\sum_{k=0}^{n}\cos^k((n+1)x)=(n+1)$$

$$D'où\lim_{x\to 0}\left(\prod_{k=1}^{n}\cos^k(kx)\times\sum_{k=0}^{n}\cos^k((n+1)x)\times\frac{(n+1)^2}{a^2}\times\frac{\left(1-\cos((n+1)x)\right)}{((n+1)x)^2}\times\frac{(ax)^2}{1-\cos(ax)}\right)=\frac{(n+1)^3}{a^2}$$

$$Par\ suite\ \left(\forall n\in\mathbb{N}^*\right)\ ;\ U_{n+1}-U_n=\frac{(n+1)^3}{a^2}$$

2) $On\ a:\ \forall k\in\{1;2;.......;n-1\}\ ;\ U_{k+1}-U_k=\frac{(k+1)^3}{a^2}\Rightarrow\sum_{k=1}^{n-1}U_{k+1}-U_k=\sum_{k=1}^{n-1}\frac{(k+1)^3}{a^2}$

$$\Rightarrow\sum_{k=1}^{n-1}U_{k+1}-\sum_{k=1}^{n-1}U_k=\sum_{k=1}^{n-1}\frac{(k+1)^3}{a^2}$$

$$\Rightarrow U_n-U_1=\frac{1}{a^2}\sum_{k=1}^{n-1}(k+1)^3$$

$$\Rightarrow U_n=\frac{1}{a^2}\sum_{k=1}^{n-1}(k+1)^3+U_1$$

$$Et\ U_1=\lim_{x\to 0}g_1(x)$$

$$=\lim_{x\to 0}\frac{1-\cos(x)}{1-\cos(ax)}$$

$$=\lim_{x\to 0}\frac{1}{a^2}\times\frac{1-\cos(x)}{x^2}\times\frac{(ax)^2}{1-\cos(ax)}=\frac{1}{a^2}$$

$$Donc\ U_n = \frac{1}{a^2}\sum_{k=1}^{n-1}(k+1)^3 + \frac{1}{a^2}$$
$$= \frac{1}{a^2}\left(\sum_{k=1}^{n-1}(k+1)^3 + 1\right)$$
$$= \frac{1}{a^2}\left(\sum_{k=1}^{n}k^3\right)$$
$$= \frac{1}{a^2}\times\frac{n(n+1)^2}{2}$$

Exercice 2 :

Soit $a \in [0;+\infty[$

1) *Montrer que* $\arctan\left(\sqrt{a}+\sqrt{a+1}\right) = \frac{\pi}{4} + \frac{1}{2}\arctan\left(\sqrt{a}\right)$

2) *Déduire que :* $\tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}$

Correction

1) *Soit* $a \in [0;+\infty[$ *; on pose :* $\sqrt{a} = \tan(\alpha)$ *où* $\alpha \in \left[0;\frac{\pi}{2}\right[$

$$\sqrt{a} = \tan(\alpha) \Leftrightarrow a = \tan^2(\alpha)$$
$$\Leftrightarrow a+1 = 1+\tan^2(\alpha)$$
$$\Leftrightarrow \sqrt{a+1} = \sqrt{1+\tan^2(\alpha)}$$
$$\Leftrightarrow \sqrt{a+1} = \sqrt{\frac{1}{\cos^2(\alpha)}}$$
$$\Leftrightarrow \sqrt{a+1} = \frac{1}{\cos(\alpha)} \quad \left(car\ \alpha \in \left[0;\frac{\pi}{2}\right[\Rightarrow \cos(\alpha) > 0\right)$$

$$Et\ \cos(\alpha) = \frac{1-\tan^2\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}\ ;\ \tan(\alpha) = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)}$$

$$Donc\ \sqrt{a+1} = \frac{1+\tan^2\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)}$$

D'où $\sqrt{a}+\sqrt{a+1}=\tan(\alpha)+\dfrac{1+\tan^2\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)}$

$$=\frac{2\tan\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)}+\frac{1+\tan^2\left(\frac{\alpha}{2}\right)}{1-\tan^2\left(\frac{\alpha}{2}\right)}$$

$$=\frac{\left(1+\tan\left(\frac{\alpha}{2}\right)\right)^2}{\left(1-\tan\left(\frac{\alpha}{2}\right)\right)\left(1+\tan\left(\frac{\alpha}{2}\right)\right)}$$

$$=\frac{1+\tan\left(\frac{\alpha}{2}\right)}{1-\tan\left(\frac{\alpha}{2}\right)}$$

$$=\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\alpha}{2}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\alpha}{2}\right)}$$

$$=\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)$$

D'où $\arctan\left(\sqrt{a}+\sqrt{a+1}\right)=\arctan\left(\tan\left(\frac{\alpha}{2}+\frac{\pi}{4}\right)\right)$

$$=\frac{\alpha}{2}+\frac{\pi}{4}$$

$=\frac{\pi}{4}+\frac{1}{2}\arctan\left(\sqrt{a}\right)$ *(car* $\sqrt{a}=\tan(\alpha)\Rightarrow\alpha=\arctan\left(\sqrt{a}\right)$*)*

Donc $\arctan\left(\sqrt{a}+\sqrt{a+1}\right)=\frac{\pi}{4}+\frac{1}{2}\arctan\left(\sqrt{a}\right)$

2) *On a :* $\left(\forall a\in[0;+\infty[\right)$; $\arctan\left(\sqrt{a}+\sqrt{a+1}\right)=\frac{\pi}{4}+\frac{1}{2}\arctan\left(\sqrt{a}\right)$

Pour $a=3$ *; on obtient :*

$$\arctan\left(\sqrt{3}+\sqrt{4}\right)=\frac{\pi}{4}+\frac{1}{2}\arctan\left(\sqrt{3}\right)\Leftrightarrow\arctan\left(\sqrt{3}+\sqrt{4}\right)=\frac{\pi}{4}+\frac{1}{2}\times\frac{\pi}{3}$$

$$\Leftrightarrow \arctan\left(\sqrt{3}+2\right)=\frac{5\pi}{12}$$

$$\Leftrightarrow \tan\left(\frac{5\pi}{12}\right)=\sqrt{3}+2 \quad (Car\ \frac{5\pi}{12}\in\left[0;\frac{\pi}{2}\right[\ )$$